Functional analysis
List 3

Ex 1. Let $K \in L_2([0,1] \times [0,1])$. Prove, that operator $T : L_2[0,1] \to L_2[0,1]$ defined by the formula

$$(Tx)(s) = \int_0^1 K(s,t)x(t)dt$$

is continuous.

Ex 2. Let $\mathcal{D}(A) = \{x = (x(n)) \in l_2 : \sum_{n=1}^{\infty}(nx(n))^2 < \infty\}$. Check whether an operator $A : \mathcal{D}(A) \to l_2$ given by the formula

$$A(x(1), x(2), ...) = (x(1), 2x(2), 3x(3), ...)$$

is continuous?

Ex 3. Let $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ and $\|x\|_1 = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |x'(t)|$. Prove that

a) operator $\frac{d}{dt} : (C^{(1)}[0,1], \|\cdot\|_1) \to (C[0,1], \|\cdot\|_\infty)$ is bounded and find its norm.

b) operator $\frac{d}{dt} : (C^{(1)}[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$ is unbounded.

Ex 4. Find the norm of an operator $A : l_p \to l_p$ given by the formula
a) $Ax = (x(1), x(2), ..., x(n), 0, 0, ...)$
b) $Ax = (x(1), x(3), x(5), x(7), ...)$
c) $Ax = (x(2), x(3), x(4), ...)$
d) $Ax = (\frac{1}{3}x(1), \frac{1}{3^2}x(2), ..., \frac{1}{3^n}x(n), ...)$
e) $Ax = (2x(1), \frac{9}{4}x(2), ..., (\frac{n+1}{n})^n x(n), ...)$

Ex 5. Find the norm of functional $Tx = \sum_{i=1}^{n} c_i x(t_i)$ acting on the space of continous real functions $C[a,b]$; $c_i \in \mathbb{R}$, $t_i \in [a,b]$. Consider the case when $C[a,b]$ is the space continous complex functions; $c_i \in \mathbb{C}$, $t_i \in [a,b]$.

Ex 6. Find a norm of a linear operator $A : C[a,b] \to C[a,b]$ given by $(Ax)(t) = tx(t)$, for $x \in C[a,b]$.

Ex 7. Calculate a norm of a functional $Tx = \int_{-1}^{1} tx(t)dt$ in $C[-1,1]$.

Ex 8. Calculate a norm of a functional $Tx = \int_{-1}^{1} tx(t)dt$ in $L_p(-1,1)$, for $p \in [1,\infty]$.

Ex 9. Find a norm of an operator $(Ax)(t) = tx(t)$ where

a) $A : C[-1,1] \to L_p(-1,1)$, $1 \le p < \infty$,

b) $A : L_p(-1,1) \to L_1(-1,1)$, $1 \le p < \infty$,

c) $A : L_{p_2}(-1,1) \to L_{p_1}(-1,1)$, $1 \le p_1 < p_2 < \infty$.

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Teaching aids

Definition 1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. We say that a map $A : X \to Y$ is a *linear operator* if

$$A(x+y) = Ax + Ay, \qquad A(\lambda x) = \lambda Ax$$

for all $x, y \in X$ and every number λ. We say that A is *bounded* if there exists a number $M > 0$ such that

$$\|Ax\|_Y \leq M\|x\|_X \qquad \text{for all } x \in X.$$

Proposition 2. A linear operator $A : X \to Y$ is a bounded if and only if it is conitinuous

Ex 10. Prove Proposition 2.

Definition 3. If $A : X \to Y$ is a bounded linear operator we define its norm $\|A\|$ in the following way

$$\|A\| = \inf\{M : \|Ax\|_Y \leq M\|x\|_X \ \text{ for all } \ x \in X\}$$

Proposition 4. If $A : X \to Y$ is a bounded operator then

$$\|A\| = \sup_{\|x\|_X = 1} \|Ax\|_Y.$$

Ex 11. Prove Proposition 4.

Ex 12. Show that the set $L(X, Y)$ of all bounded linear operators from X to Y with pointwise operations and operator norm as a norm forms a normed space.

Ex 13. Show that if Y is a Banach space then $L(X, Y)$ is a Banach space.